Matb41 Midterm Notes

Weeks 1-6

I. Lines

1. Def : A line in R^n is decided by: 1. 2 points
   2. A point and a direction

2. Lines in R^2:

- Has the eqn $Ax + By = C$

- 2-Point Eqn:
  Given 2 points, (x_1, y_1) and (x_2, y_2), we can use the eqn $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$ to find the line that passes through both points.

  Note: We can rewrite the formula above to
  $$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1$$

- Vector Eqn:
  A way to represent a line using a point and a direction.

  Given a point P, we can find its position vector $\vec{P}$. Furthermore, let $\vec{v}$ be a vector.

  $$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\vec{P}} + t \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{v}}, \ t \in R$$

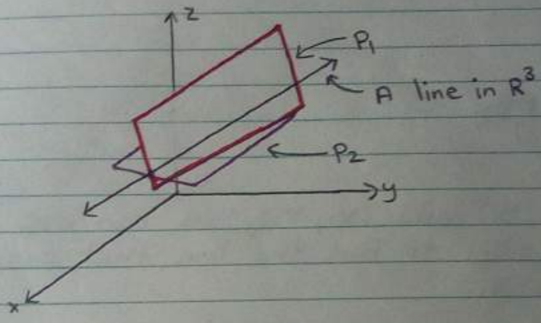
- Parametric Eqn:

  $$\begin{cases} x = x_0 + at \\ y = y_0 + bt \end{cases}, \ t \in R$$

  Note: Vector and parametric eqns are NOT unique.

3. Lines in R^3:

- A line in R^3 is the intersection of 2 non-parallel planes.



- Vector and parametric eqns in R^3 are the same as in R^2.

- Symmetric Eqn of a Line:

$$x = x_0 + at \rightarrow t = \frac{x - x_0}{a}$$

$$y = y_0 + bt \rightarrow t = \frac{y - y_0}{b}$$

$$z = z_0 + ct \rightarrow t = \frac{z - z_0}{c}$$

$\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c} = t$ is the symmetric eqn of a line.

Note: Symmetric eqns are Not unique.

4. Examples:

Q1 For each of the following lines, write both its vector and parametric eqns.

a) The line that passes thru $(1,2,4)$ and is in the direction of $[5,-3,1]$.

Soln:

$P = (1,2,4)$
$\vec{v} = [5,-3,1]$

Symm:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

Parametric:

$$\begin{cases} x = 1 + 5t \\ y = 2 - 3t \\ z = 4 + t \end{cases}, t \in \mathbb{R}$$

b) The line is perpendicular to the lines $r_1(t) = (4t, 1+2t, 3t)$ and $r_2(s) = (-1+s, -7+2s, -12+3s)$ and passes thru the point of intersection of $r_1(t)$ and $r_2(s)$.

Soln:
Both lines are written in vector form.
$r_1(t) = (0,1,0) + [4,2,3]t.$
$r_2(s) = (-1,-7,-12) + [1,2,3]s.$

To get a line perpendicular to both $r_1(t)$ and $r_2(s)$, we need to cross product their vectors.

$$[4,2,3] \times [1,2,3] = [0,-9,6].$$

This is the directional vector of the line.

To find their P. O. I,

$4t = -1+s$ ①
$1+2t = -7+2s$ ②
$3t = -12+3s$ ③

$t = -4+s$ From 3
$4t = 4(-4+s)$ Subbing into 1
$= -16+4s$
$-16+4s = -1+s$
$s = 5$

$-1+5 = 4$
$-7+2(5) = 3$
$-12+3(5) = 3$

$(4,3,3)$ is the POI,

Symm:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} + r\begin{bmatrix} 0 \\ -9 \\ 6 \end{bmatrix}, \; r \in R$$

Parametric:

$$\begin{cases} x = 4 \\ y = 3-9r, \quad r \in R \\ z = 3+6r \end{cases}$$

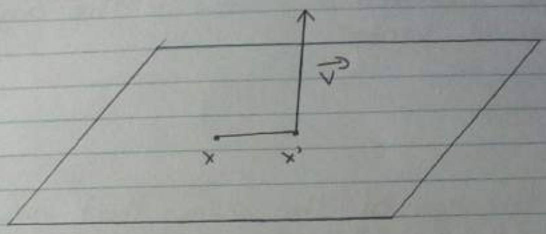
Q2 Find a symm eqn of a line that goes thru $(1,1,0)$ and $(0,4,7)$.

Soln:
$\vec{v} = [1,1,0] - [0,4,7]$
$= [1,-3,-7]$

$$\frac{x-1}{1} = \frac{y-1}{-3} = \frac{z}{-7} = t$$

2. Planes:

1. Def: A plane in R^n may be decided by a point on the plane and a vector that is orthogonal to the plane.



$\vec{v}$ is perpendicular to the plane.
$x' - x$ is on the plane.

$(x'-x)\cdot\vec{v} = 0$
$\rightarrow [x_1' - x_1, x_2' - x_2, \ldots x_n' - x_n] \cdot [v_1, v_2, \ldots v_n] = 0$
$\rightarrow v_1(x_1' - x_1) + v_2(x_2' - x_2) + \ldots + v_n(x_n' - x_n) = 0$
$\rightarrow v_1 x_1' + v_2 x_2' + \ldots + v_n x_n' = \underbrace{v_1 x_1 + v_2 x_2 + \ldots + v_n x_n}_{\text{This is a constant.}}$

$v_1 x_1' + v_2 x_2' + \ldots + v_n x_n' = v_1 x_1 + v_2 x_2 + \ldots + v_n x_n$ is the eqn of a plane.

In R^3, this can be generalized as $Ax + By + Cz = D$.

2. Intersections Between 2 Planes:

- If the planes are non-parallel, then the intersection is a line.

- If the planes are parallel, there is no intersection.

3. Angle Between 2 Planes:

- 2 planes are parallel if their normal vectors are parallel.
- The angle between 2 planes is the angle between their normal vectors.

4. Examples:

Q1 Find the eqn of the plane that goes through the points $(1, 2, 5)$, $(5, 4, 8)$ and $(2, 4, 8)$.

Soln:

$\vec{V_1} = (5, 4, 8) - (2, 4, 8)$
$= (3, 0, 0)$

$\vec{V_2} = (1, 2, 5) - (2, 4, 8)$
$= (-1, -2, -3)$

$\vec{V_1} \times \vec{V_2} = [0, 9, -6]$

$9y - 6z = D$

Subbing in the point $(1, 2, 5)$, we get:

$D = 9(2) - 6(5)$
$= -12$

$\therefore$ The eqn of the plane is $9y - 6z = -12$

Q2 Find the angle between the 2 planes
1. $5x - 3y + 2z = 11$
2. $x + 3y + 2z = 5$

Soln:

$\vec{V_1} = [5, -3, 2]$
$\vec{V_2} = [1, 3, 2]$

$$\theta = \cos^{-1}\left(\frac{\vec{V_1} \cdot \vec{V_2}}{\|\vec{V_1}\| \, \|\vec{V_2}\|}\right) = \cos^{-1}\left(\frac{0}{\sqrt{38}\sqrt{14}}\right) = \frac{\pi}{2} \quad \theta = \frac{\pi}{2}$$

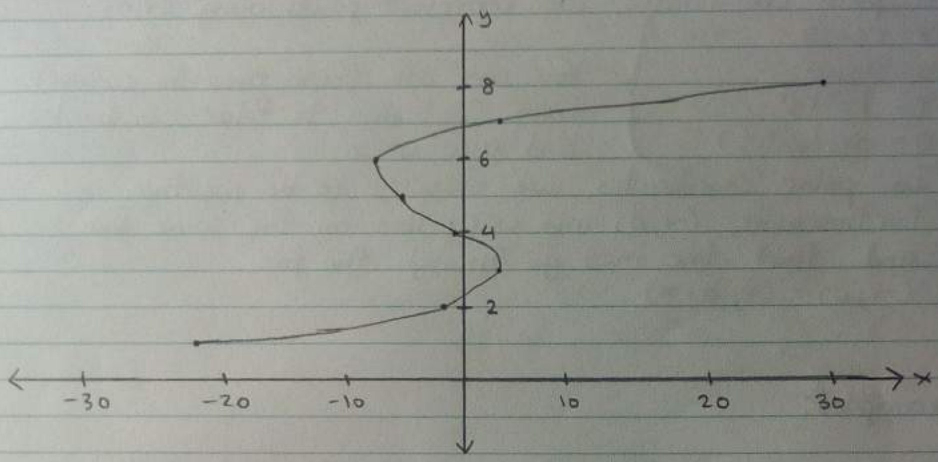
3. Curves:

1. Examples:

Q1 Sketch the curve $x = t^3 - 4t^2 + 2$, $y = t + 3$, $-2 \le t \le 5$

Soln:

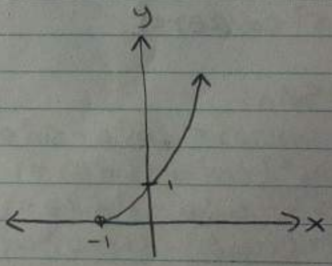
t	-2	-1	0	1	2	3	4	5
x	-22	-3	2	-1	-6	-7	2	27
y	1	2	3	4	5	6	7	8



2. Eliminate the parameters to find a Cartesian eqn of the curve.

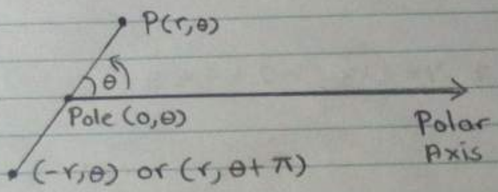
$x = e^t - 1$
$y = e^{2t}$

$e^t = x + 1$
$y = (e^t)^2$
$= (x+1)^2$



Note: $x > -1$ because $e^t > 0$.

4. Polar Coordinates:



1. Definition:

- In polar coordinates, we represent points using (r,θ).
- $$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\left(\tfrac{y}{x}\right) \end{aligned}\right\}$$ You can use these eqns to convert from Cartesian to Polar coordinates and vice versa.
- In polar coordinates, we allow r to be negative. Furthermore, $(-r,\theta)$ and (r,θ) lies on the same line and that line must go through the pole.
  $(-r,\theta) = (r, \theta+\pi)$.

2. Examples:

Q1 Convert the following polar equation to cartesian equation.

$$r^2\cos(2\theta)=1$$

Soln:

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1$

$\rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1$

$\rightarrow (r\cos\theta)^2 - (r\sin\theta)^2 = 1$

$\rightarrow x^2 - y^2 = 1$

Q2 Convert the cartesian equation to polar equation.

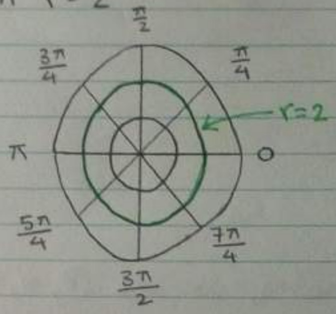
$xy = 4$

Soln:

$x = r\cos\theta$
$y = r\sin\theta$

$\rightarrow r^2(\sin\theta\cos\theta) = 4$
$\rightarrow r^2(\frac{1}{2})(2\sin\theta\cos\theta) = 4$
$\rightarrow r^2(\frac{1}{2})(\sin 2\theta) = 4$
$\rightarrow r^2 = 8\csc 2\theta$

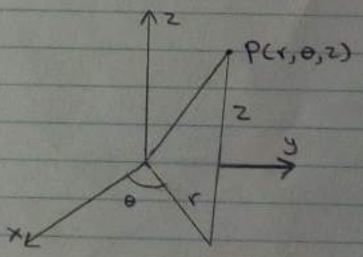
Q3 Graph $r = 2$



5. Cylindrical Coordinates:

1. Definition:
   - When we extend polar coordinates from R^2 to R^3, we get cylindrical coordinates.
   - Cylindrical coordinates uses (r, θ, z).

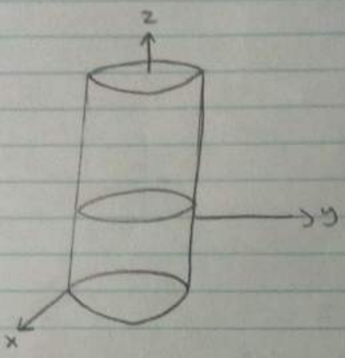
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$r = \sqrt{x^2+y^2}$
$\theta = \arctan(\frac{y}{x})$



2. Examples:

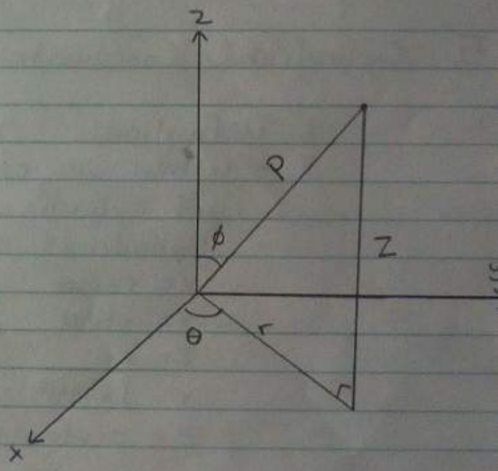
Q1 Sketch $r=2$ in cylindrical coordinates.

Soln:
In R^2, $r=2$ is a circle.
In R^3, $r=2$ is a cylinder.



6. Spherical Coordinates:

1. Definition:
- In <u>spherical</u> coordinates, we use (ρ, θ, ϕ).
- $\rho = \sqrt{x^2+y^2+z^2}$
  $r = \rho \sin\phi$
  $x = r\cos\theta = \rho \sin\phi \cos\theta$
  $y = r\sin\theta = \rho \sin\phi \sin\theta$
  $z = \rho\cos\phi$
  $\theta = \arctan\left(\frac{y}{x}\right)$
  $\phi = \arctan\left(\frac{r}{z}\right) = \arccos\left(\frac{z}{\rho}\right)$



2. Examples:

Q1 Convert the cartesian coordinates, $(2,3,6)$, into spherical coordinates.

Soln:

$$\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2} \\ &= \sqrt{2^2+3^2+6^2} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

$$\begin{aligned} \theta &= \arctan\left(\frac{y}{x}\right) \\ &= \arctan\left(\frac{3}{2}\right) \end{aligned}$$

$$\begin{aligned} \phi &= \arccos\left(\frac{z}{\rho}\right) \\ &= \arccos\left(\frac{6}{7}\right) \end{aligned}$$

7. Vector Functions:

1. Definition

- A vector-valued function $f: R^n \rightarrow R^m$ is a rule or process that assigns each input x in R^n to its corresponding output y in R^m. $m>1$.

- If $m=1$, it is called a scalar-valued function or real-valued function.

8. Graphs of Functions:

1. Definition:

- Let $f: U \subset R^n \rightarrow R$. The graph of f is defined to be the subset of R^{n+1} consisting of the points $(x_1, x_2, \dots x_n, f(x_1, \dots x_n))$ in R^{n+1} for $(x_1, x_2, \dots x_n)$ in U.

9. Level Sets:

1. Definition:

- Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ and let $k \in \mathbb{R}$. The level set of f at value k is defined to be the set of those points $x \in U$ at which $f(x) = k$.

If $n=2$, we have level curve/level contour,
If $n=3$, we have level surfaces.

2. Examples:

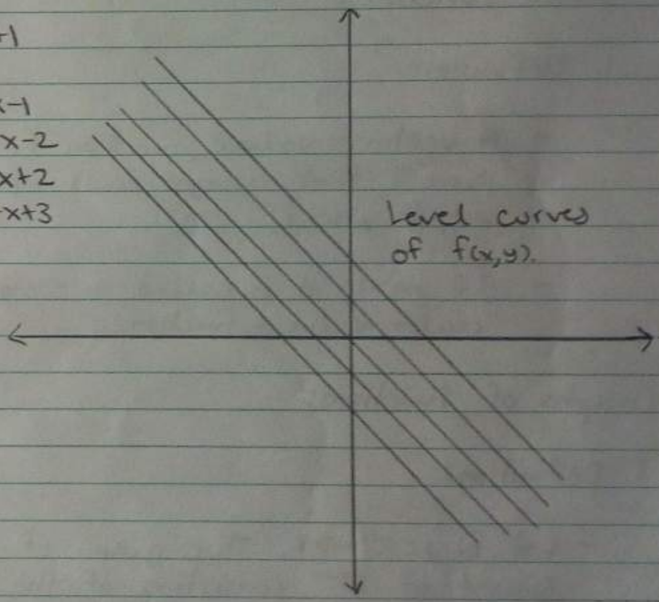
Q1 Draw the level curves for the function $f(x,y) = 1-x-y$.

Soln:

Let $k = 1-x-y$, $k \in \mathbb{R}$

Now, we choose various values for k and solve for x and y.

$k=0 \to y=-x+1$
$k=1 \to y=-x$
$k=2 \to y=-x-1$
$k=3 \to y=-x-2$
$k=-1 \to y=-x+2$
$k=-2 \to y=-x+3$



10. Open Sets:

1. Definition:

- Let $U \subset \mathbb{R}^n$. U is an open set if for every point x_0 in U, there exists some $r>0$ s.t. $D_r(x_0)$ is contained in U.

I.e. For any object to be an open set, any arbitrary point within that set must be able to form a smaller open set.

In $\mathbb{R}^1$, an open set is an open interval.
In $\mathbb{R}^2$, an open set is an open disk.
In $\mathbb{R}^3$, an open set is an open ball.

2. Open Disk/Open Ball:

- An open disk/open ball of radius r and center x_0 is the set of all points x s.t. $\|x_0 - x\| < r$. This is denoted as $D_r(x_0)$.

3. Proving Something is an Open Set:

- To prove that something is an open set, we have to prove that any arbitrary point within it can form a smaller open set.

4. Thm: $D_r(x_0)$ is an open set:

Proof:
Let y be an arbitrary point in $D_r(x_0)$.
Then, $\|y - x_0\| < r$.
Let $s = r - \|y - x_0\|$, $s > 0$
Let $D_s(y) = \{x \in \mathbb{R}^n \mid \|x-y\| < s\}$
$\forall x \in D_s(y)$, we need to show that $\|x - x_0\| < r$.

$$\begin{aligned} \|x - x_0\| &= \|x + y - y - x_0\| \\ &\leq \|x - y\| + \|y - x_0\| \quad \text{Triangle Inequality} \\ &< \delta + \|y - x_0\| \\ &= r \end{aligned}$$

$\therefore$ $D_r(x_0)$ is an open set.

II. Method of Sections:

1. We take the intersection of the graph and the z-axis and graph the different sections.

2. Examples:

Q1 Find the method of sections for the function $z = x^2 + y^2$.

Solns:

1. We set $x = 0$. Then, we get the section of the function that is on the yz plane $\cap \{(x,y,z) \mid x = 0, z = y^2\}$

2. We set $y = 0$. Now, we get xz plane $\cap \{(x,y,z) \mid y = 0, z = x^2\}$.

I.e. Set $x = 0$ and graph/simplify the function.
Set $y = 0$ and graph/simplify the function.

12. Delta - Epsilon Proof:

1. Informal Definition of Limits:
   - Let $a = (a_1, a_2, \ldots a_n)$ and $x = (x_1, x_2, \ldots x_n)$ be points in R^n. Let $f: R^n \to R$. L is called the limit of f as x approaches a if $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a.

2. Delta- Epsilon Proof Definition:

$$\lim_{x \to a} f(x) = L \text{ if}$$

$$\forall \varepsilon > 0, \exists d > 0 \text{ s.t. if } 0 < \| x - a \| < d \text{ then } |f(x) - L| < \varepsilon.$$

3. Examples:

Q1 Use the definition of a limit to prove that
$$\lim_{(x,y) \to (0,0)} xy = 0$$

Soln:
$\forall \varepsilon > 0, \exists d > 0$ s.t. if $0 < \| (x,y) - (0,0) \| < d$ then $|xy - 0| < \varepsilon$.

$$\| (x,y) \| = \sqrt{x^2 + y^2} < d$$
$$x^2 + y^2 < d^2$$
$$(x+y)^2 = x^2 + 2xy + y^2 \geq 0$$
$$x^2 + y^2 \geq 2xy$$
$$\geq 2|xy|$$
$$\frac{x^2+y^2}{2} \geq |xy|$$

Choose $\frac{d^2}{2} = \varepsilon \to d = \sqrt{2\varepsilon}$

Proof:
$$|xy| \leq \frac{x^2+y^2}{2}$$
$$< \frac{d^2}{2}$$
$$= \varepsilon, \text{ as wanted}$$

13. Paths of Limits:

1. Definition:

   In multi-variable limits, we could approach a point from several directions. For a limit to exist, the function must be approaching the same value regardless of the path it takes.

   I.e. If x approaches point a along path A results in $f(x) = L$ and x approaches point a along path B results in $f(x) = M$, and $L \neq M$, then the limit DNE.

   Note: We only use this to prove a limit DNE.

2. Examples:

   Q1 Disprove the limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ exists.

   Solution:

   1. Along the path $y=0$, we get
      $\lim\limits_{x\to 0} \frac{x^2}{x^2} = 1$

   2. Along the path $x=0$, we get
      $\lim\limits_{y\to 0} \frac{-y^2}{y^2} = -1$

   Since $1 \neq -1$, the limit DNE.

14. Continunity:

1. Informal Definition:
Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function with domain U. Let $x_0 \in U$. We say f is cont at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$.

$\lim_{x \to x_0} f(x) = f(x_0)$ means

1. $x_0 \in \text{Domain}(f)$
2. $\lim_{x \to x_0} f(x) = L$
3. $f(x_0) = L$

If f doesn't satisfy any of these reqs, then f is not cont at x_0.

2. Formal Definition:
$\lim_{x \to x_0} f(x) = f(x_0)$ means

$\forall \varepsilon > 0 \ \exists d > 0$ s.t. if $\|x - x_0\| < d$ then $|f(x) - f(x_0)| < \varepsilon$.

3. Examples:

Q1 Is $f(x,y) = \dfrac{x^2 - y^2}{x+y}$ cont at $(0,0)$?

Soln:
$f(x,y)$ is not cont at $(0,0)$ because there is a hole at $(0,0)$. This means that $(0,0) \notin \text{Dom}(f)$.

Note: $(0,0)$ is a removal discontinunity because $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = 0$. The limit exists but $f(0,0)$ doesn't.

Q2. Is $3x^2y+\sqrt{xy}$ cont at $(1,2)$?

Soln:

1. $(1,2) \in Dom(f)$

2. $\lim_{(x,y)\to(1,2)} 3x^2y+\sqrt{xy} = 6+\sqrt{2}$

3. $f(1,2) = 6+\sqrt{2}$

$\therefore$ $3x^2y+\sqrt{xy}$ is cont at $(1,2)$.

Q3. Is $f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

cont at $(0,0)$?

Soln:

1. $(0,0) \in Dom(f)$

2. Along the path $y=x$, we get:

$$\lim_{x\to 0} \frac{(2x)^2}{2x^2} = 2$$

Along the path $y=-x$, we get:

$$\lim_{x\to 0} \frac{0}{2x^2} = 0$$

Since $2 \neq 0$, the limit DNE.

$\therefore$ f is not cont at $(0,0)$.

4. Properties of Continuity:

1. Let $f(x)$ and $g(x)$ be cont at x_0 and let c be a constant. Then,

   1. $f(x) \pm g(x)$
   2. $cf(x)$
   3. $f(x)g(x)$
   4. $\frac{f(x)}{g(x)}$, $g(x) \neq 0$

   are cont at x_0.

2. Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ with $f(x) = (f_1(x), f_2(x), \dots f_m(x))$. f is cont at x_0 iff $f_1, f_2, \dots f_m$ are cont at x_0.

3. Let $g: U_1 \subset \mathbb{R}^n \to \mathbb{R}^m$
   Let $f: U_2 \subset \mathbb{R}^m \to \mathbb{R}^p$
   Suppose that $g(U_1) \subset U_2$, s.t. $f \circ g$ is defined on U_1. If g is cont at x_0 and if f is cont at $g(x_0)$, then $f \circ g$ is cont at x_0.

4. Trig, polynominal, and exponential functions are cont on their domain.

5. If $f(x,y)$ is cont on (a,b), then you can plug (a,b) into $f(x,y)$ to find $\lim_{(x,y)\to(a,b)} f(x,y)$.

6. Example:

   Q1 Show that $\sin(x+y)$ is cont everywhere in $\mathbb{R}^2$.

   Soln:
   $x+y$ is a polynominal and $\sin(t)$ is a trig function. Both are cont everywhere in $\mathbb{R}^2$, so $\sin(x+y)$ is also cont everywhere in $\mathbb{R}^2$ by composition property.

15. Techniques of Limits:

1. Delta- Epsilon Proof:
   - We can use this to prove a limit exists.

2. Approaching the point from various directions:
   - Used to prove a limit DNE.

3. Plugging in the point:
   - This can only be used if $f(x,y)$ is cont at the point.

4. Substitution:
   - This can turn a function of multiple variables into a function of 1 variable. Then, you may use l'hopital's rule or another rule on it.

5. Taylor Series:
   - We can substitute some functions for their Taylor Series counterpart.
   - $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
   - $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
   - $\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$

6. Squeeze Thm:
   - Used mainly with functions that have sin or cos in it.

7. Pay attention to the degree of the numerator and denominator. Polynominals with higher degrees reach 0 faster.

8. Use the fact that:

1. $|x| \leq |x+y| \leq |x+y+z|$
2. $|x-a| \leq \sqrt{(x-a)^2+(y-b)^2}$
3. $\left(\frac{x^2}{x^2+y^2}\right) \leq 1$

9. Examples:

Q1 Use the definition to prove that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$.

Soln:

$\forall \varepsilon > 0, \exists d > 0$ s.t. if $0 < \|(x,y)-(0,0)\| < d$ then $\left|\frac{xy^2}{x^2+y^2} - 0\right| < \varepsilon$

$$\begin{aligned}\left|\frac{xy^2}{x^2+y^2}\right| &\leq |x|\left(\frac{y^2}{x^2+y^2}\right)\\ &\leq |x|\\ &\leq |x+y|\\ &= \sqrt{x^2+y^2}\end{aligned}$$

Choose $d = \varepsilon$.

Proof:

$$\begin{aligned}&0 < \|(x,y)\| < d\\ &\sqrt{x^2+y^2} < d\\ &|x+y| < d\\ &|x| < d\\ &|x|\left(\frac{y^2}{x^2+y^2}\right) < d\end{aligned}$$

$$\begin{aligned}\left|\frac{xy^2}{x^2+y^2}\right| &< d\\ &= \varepsilon, \text{ as wanted}\end{aligned}$$

Q2 Evaluate $\lim_{(x,y)\to(1,2)} xy^2$.

Soln:

Since xy^2 is cont at $(1,2)$, we can plug it in.

$\lim_{(x,y)\to(1,2)} xy^2 = (1)(2)^2 = 4$

Q3 Evaluate $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}$

Soln:

Let $r = x+y$

$\lim_{r\to 0} \frac{\sin(r)}{r} = 1$

$\therefore \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = 1$

Q4 Evaluate $\lim_{(x,y)\to(0,0)} (x)(\sin(\frac{1}{x+y}))$

Soln:

We know that $-1 \leq \sin(\frac{1}{x+y}) \leq 1$

$\lim_{x\to 0} -x = 0$

$\lim_{x\to 0} x = 0$

$\therefore$ By ST, $\lim_{(x,y)\to(0,0)} (x)(\sin(\frac{1}{x+y})) = 0$

Q5 Evaluate $\lim_{(x,y)\to(1,1)} \frac{\ln(xy)}{xy-1}$

Soln:

Recall that $\ln(1)$ is 0.

$\ln(xy) = (xy-1) - \frac{1}{2}(xy-1)^2 + \frac{1}{3}(xy-1)^3 - \dots$

$$\lim_{(x,y)\to(1,1)} \frac{(xy-1) - \frac{1}{2}(xy-1)^2 + \frac{1}{3}(xy-1)^3 - \dots}{xy-1}$$

$$= \lim_{(x,y)\to(1,1)} 1 - \frac{1}{2}(xy-1) + \dots$$

$$= 1$$

16. Properties of Limits:

Let $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$ and c be a constant.

1. $\lim_{x\to a} c = c$

2. $\lim_{x\to a} cf(x) = cL$

3. $\lim_{x\to a} (f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L+M$

4. $\lim_{x\to a} (f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = LM$

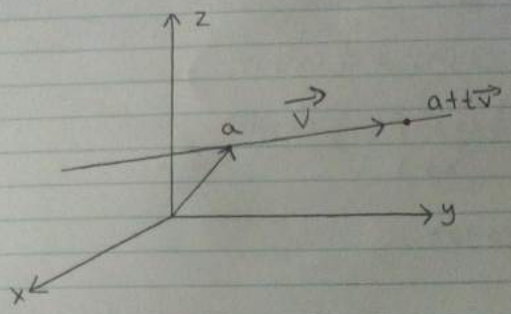
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, $\lim_{x\to a} g(x) \neq 0 = \frac{L}{M}$, $M \neq 0$

6. $\lim_{x\to a} (f(x))^{\frac{m}{n}} = L^{\frac{m}{n}}$, $n \neq 0$

17. Differentiation:

1. Definition:

- Let $z = f(x)$. To find the rate of change in f at a point, a, along the line $a + t\vec{v}$, $t \in \mathbb{R}$.



$$\Delta x = (a + t\vec{v}) - a$$
$$= t\vec{v}$$

$$\Delta f = f(a + t\vec{v}) - f(a)$$

$$\frac{\Delta f}{\Delta x} = \lim_{t \to 0} \frac{f(a + t\vec{v}) - f(a)}{t\|\vec{v}\|}$$

2. Directional Derivatives:

1. Definition:

- Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. The directional derivative of f at "a" in direction $\vec{v}$, denoted by $D_v(f(a))$.

$$D_v(f(a)) = \lim_{t \to 0} \frac{f(a + t\vec{v}) - f(a)}{t\|\vec{v}\|}$$

Note: If $\vec{v}$ is a unit vector, then the formula is $\lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a)}{t}$.

Note: If $\vec{v} = e_i$, $i = 1, 2, \ldots n$, $D_{e_i}(f(a))$ is denoted as $\frac{\partial f}{\partial x_i}(a)$ and is called the partial derivative of f with respect to x_i at "a".

I.e. Partial derivatives represents the rate of change of f as we vary x_i and hold the other variables constant.

2. Examples

Q1 Let $f(x,y,z) = x^2 - 2y + 3z^3$.
Find the directional derivative of f at $(0,1,0)$ in the direction of

a) $\vec{v} = [1,1,1]$

Soln:

$$\lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a)}{t\|\vec{v}\|}$$
$$= \lim_{t \to 0} \frac{f((0,1,0) + t[1,1,1]) - f(0,1,0)}{(t)(\sqrt{3})}$$
$$= \lim_{t \to 0} \frac{f(t, 1+t, t) - f(0,1,0)}{\sqrt{3}t}$$
$$= \lim_{t \to 0} \frac{t^2 - 2(1+t) + 3t^3 - 2}{\sqrt{3}t}$$
$$= \lim_{t \to 0} \frac{3t^3 + t^2 - 2t}{\sqrt{3}t}$$
$$= \lim_{t \to 0} \frac{3t^2 + t - 2}{\sqrt{3}}$$
$$= \frac{-2}{\sqrt{3}}$$

b) $\vec{v} = [1,0,0]$

Soln:
Since $\vec{v}$ is a unit vector, $\|\vec{v}\| = 1$.

$$\lim_{t\to 0} \frac{f(a+t\vec{v}) - f(a)}{t}$$

$$= \lim_{t\to 0} \frac{f((0,1,0) + t[1,0,0]) - f(0,1,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(t,1,0) - f(0,1,0)}{t}$$

$$= \lim_{t\to 0} \frac{t^2 - 2 - (-2)}{t}$$

$$= \lim_{t\to 0} \frac{t^2}{t}$$

$$= 0$$

Q2 Calculate the partial derivatives of $f(x,y,z) = x^2 - 2y + 3z^3$.

Soln:

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = -2$$

$$\frac{\partial f}{\partial z} = 9z^2$$

18. Differentiability:

1. Defintion:

- Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. f is diff at $a \in U$ if:
  1. The partial derivatives of f exist at a
  2. $\lim_{x \to a} \frac{\| f(x) - f(a) - D(f(a))(x-a) \|}{\|x-a\|} = 0$

$Df(a)$ is the Jacobian Matrix of f at "a" given by

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_n} \end{bmatrix}$$

or denoted as $Df(a) = \frac{\partial (f_1, f_2, \dots f_k)}{\partial (x_1, x_2, \dots x_n)}(a)$.

2. Examples:

Q1 Calculate $Df(a)$ where $f(x,y,z) = (x^2 + y\sin z, xe^y, z\cos x)$ at $(1,1,1)$.

Soln:

$f_1 = x^2 + y\sin z$

$f_2 = xe^y$

$f_3 = z\cos x$

$a = (1,1,1)$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 2x & \sin z & y\cos z \\ e^y & xe^y & 0 \\ -2\sin x & 0 & \cos x \end{bmatrix}$$

At the point $(1,1,1)$, we have:

$$\begin{bmatrix} 2 & \sin(1) & \cos(1) \\ e & e & 0 \\ -\sin(1) & 0 & \cos(1) \end{bmatrix}$$

Q2 Is the function $f(x,y) = x^{\frac{1}{3}} y^{\frac{1}{3}}$ diff at $(0,0)$?

Soln:

We have to use the limit definition of partial derivatives and the fact that $f(0,0)=0$ for this.

$$f(h,0) = (h)^{\frac{1}{3}}(0)$$
$$= 0$$

$$f(0,h) = (0)(h)^{\frac{1}{3}}$$
$$= 0$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{0}{h}$$
$$= 0$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

$$\lim_{(x,y) \to (0,0)} \frac{\left\| x^{\frac{1}{3}} y^{\frac{1}{3}} - 0 - [0,0]\begin{bmatrix} x-0 \\ y-0 \end{bmatrix} \right\|}{\|(x,y)\|}$$

$$= \infty$$

$\therefore$ $f(x,y)$ is not diff at $(0,0)$.

3. Properties/Thms of Differentiability:

- Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose that the partial derivatives of f all exist and are cont in the neighbourhood $a \in U$. Then, f is diff at $a \in U$.

- If f is diff at "a", then:
  1. All partial derivatives of f at "a" exists.
  2. f is cont at "a".

4. Properties of Derivatives:

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be diff at $a \in U$. Let c be a constant. Then:

1. cf is diff at "a" and $D(cf)(a) = c(Df(a))$.
2. $f \pm g$ is diff at "a" and $D(f \pm g)(a) = Df(a) \pm Dg(a)$
3. fg is diff at "a" and $D(fg)(a) = Df(a)g(a) + Dg(a)f(a)$
4. $\frac{f}{g}$ is diff at "a" if $g(a) \neq 0$ and

$$D\left(\frac{f}{g}\right)(a) = \frac{(Df(a))g(a) - f(a)(Dg(a))}{(g(a))^2}$$

19. Chain Rule:

1. Definition:
- Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $g: V \subset \mathbb{R}^m \to \mathbb{R}^k$ be functions s.t. f maps U to V so that $g \circ f$ is defined. Let $a \in U$ and $b \in V$. If f is diff at "a" and g is diff at "b", then $f \circ g$ is diff at "a" and $D(g \circ f)(a) = D(g(f(a)))\,Df(a)$.

2. Another way of thinking about the chain rule is:
- Suppose that "y" is a diff function of n vars, $x_1, x_2, \dots x_n$ and each x_i, $i = 1, 2, \dots, n$, is a diff function of k vars, $t_1, t_2, \dots t_k$. Then, y is a function of the vars $t_1, t_2, \dots t_k$ and $\frac{\partial y}{\partial t_j} = \frac{\partial y}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \frac{\partial y}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_j} + \dots + \frac{\partial y}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_j}$.

3. Examples:

Q1 Let $z = \sin(2x+y)$
Let $x = s^2 - t^2$
Let $y = s^2 + t^2$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

Soln:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$
$$= (2\cos(2x+y))(2s) + \cos(2x+y)(2s)$$
$$= (2\cos(2(s^2-t^2) + s^2+t^2)(2s) + \cos(2(s^2-t^2)+s^2+t^2)(2s)$$
$$= (2s)(\cos(3s^2-t^2))[2+1]$$
$$= (6s)(\cos(3s^2-t^2))$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$
$$= (2\cos(2x+y))(-2t) + \cos(2x+y)(2t)$$
$$= (2t)(\cos(2x+y))[-2+1]$$
$$= -2t(\cos(3s^2-t^2))$$

20. Tangent / Velocity Vectors:

1. Definition:
- Let c be a path defined by $c(t) = (x(t), y(t), z(t))$ and let c be diff.

The tangent vector of c at t is defined by

$$c'(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$$
$$= (x'(t), y'(t), z'(t)).$$

2. Examples:

Q1 Find the tangent vector to the path of $c(t) = (t, t^2, e^t)$ at $t=0$.

Soln:

$c'(t) = (1, 2t, e^t)$
$c'(0) = (1, 0, 1)$

21. Tangent Lines:

1. Definition:
- The tangent line to c at point $a = (x(t_0), y(t_0), z(t_0))$ is defined to be the line through "a" with a direction of $c'(t_0)$.

$$\underset{\uparrow \atop \vec{x}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{\uparrow \atop a}{\begin{pmatrix} x(t_0) \\ y(t_0) \\ z(t_0) \end{pmatrix}} + \underset{\uparrow \atop c'(t_0)}{\begin{pmatrix} x'(t_0) \\ y'(t_0) \\ z'(t_0) \end{pmatrix}} (t-t_0)$$

I.e. $\vec{x} = a + (c'(t_0))(t-t_0)$

2. Example:

Q1 Find the velocity vector of the path $c(t) = (\cos t, \sin t, t)$. Then, find the tangent line of the curve at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4})$.

Soln:
$c'(t) = (-\sin t, \cos t, 1) \leftarrow$ Velocity Vector
$a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4})$
To find t_0, we need to do:
1. $\sin(t_0) = \frac{1}{\sqrt{2}} \rightarrow t_0 = \frac{\pi}{4}$
2. $\cos(t_0) = \frac{1}{\sqrt{2}} \rightarrow t_0 = \frac{\pi}{4}$
3. $t_0 = \frac{\pi}{4}$

$t_0 = \frac{\pi}{4}$

$c'(t_0) = (-\sin(\frac{\pi}{4}), \cos(\frac{\pi}{4}), 1)$
$= (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$

$\vec{x} = a + (c'(t_0))(t - t_0)$
$\vec{x} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4}) + (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)(t - \frac{\pi}{4})$

$$\begin{cases} x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(t - \frac{\pi}{4}) \\ y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(t - \frac{\pi}{4}) \\ z = \frac{\pi}{4} + (t - \frac{\pi}{4}) \end{cases}$$

22. The Gradient is Normal to Level Surfaces:

1. Definition:
- Let $f: \mathbb{R}^3 \to \mathbb{R}$ have cont partial derivatives and let (x_0, y_0, z_0) lie on the level surface S defined by $f(x,y,z) = k$, $k \in \mathbb{R}$. Then, $\nabla f(x_0, y_0, z_0)$ is orthogonal to S.

23. Tangent Planes:

1. Definition:
- Let S be the surface containing (x,y,z) s.t. $f(x,y,z) = k$, $k \in \mathbb{R}$. The tangent plane of S at point (x_0, y_0, z_0) of S is defined by the eqn:

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

2. Example:
Q1 Find the eqn of the plane tangent to the surface defined by $3xy + z^2$ at $(1,1,1)$.

Soln:
$f(x,y,z) = 3xy + z^2$
$\nabla f = [3y, 3x, 2z]$
At the point $(1,1,1)$, $\nabla f = (3,3,2)$.
$(3,3,2) \cdot (x-1, y-1, z-1) = 0$
$3x + 3y + 2z = 8$

3. Tangent Planes in R^2:

1. Definition:
- Let $f: R^2 \to R$. Let c be a level curve containing (x,y) s.t. $f(x,y)=k$, $k \in R$. Then, $\nabla f(x_0, y_0)$ is orthogonal to c for any point (x_0, y_0) on c.

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0$$

24. Linear Approximation:

1. Definition:
- Let $f: R^2 \to R$ be diff at (x_0, y_0). The linear approximation of f at (x_0, y_0) is defined by

$$L(x,y) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

2. Example: Find the linear approximation to the function $f(x,y) = \sin(xy)$ at $(1, \frac{\pi}{3})$.

Soln:

$$f(1, \tfrac{\pi}{3}) = \sin(\tfrac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

$$L(x,y) = f(1, \tfrac{\pi}{3}) + \left[\frac{\partial f}{\partial x}(1, \tfrac{\pi}{3})\right](x-1) + \left[\frac{\partial f}{\partial y}(1, \tfrac{\pi}{3})\right](y - \tfrac{\pi}{3})$$

$$= \frac{\sqrt{3}}{2} + \left(\frac{\pi}{3}\right)\left(\cos(\tfrac{\pi}{3})\right)(x-1) + (1)\left(\cos(\tfrac{\pi}{3})\right)(y - \tfrac{\pi}{3})$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{6}(x-1) + \frac{1}{2}(y - \tfrac{\pi}{3})$$

25. Directional Derivatives With Linear Approx:

1. Definition:
   - Consider a path along direction $\vec{v}$ that passes through point a. The rate of change of f in the direction $\vec{v}$ is given by $D_{\vec{v}} f(a)$.

$$D_{\vec{v}} f(a) = \lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a)}{t\|\vec{v}\|}$$

However, we can approx $f(a+t\vec{v})$.

$$L(a+t\vec{v}) = f(a) + D_{\vec{v}} f(a) \cdot \|\vec{v}\|$$
$$= f(a) + \frac{\nabla f \cdot \vec{v}}{\|\vec{v}\|}$$

Note:
1. $D_{\vec{v}} f(a) = \nabla f \cdot \frac{\vec{v}}{\|\vec{v}\|}$ if $\vec{v}$ is NOT a unit vector
2. $D_{\vec{v}} f(a) = \nabla f \cdot \vec{v}$ if $\vec{v}$ is a unit vector.

26. Finding and Sketching Domains:

1. Example

Sketch the domain of $f(x,y) = \frac{1}{\sqrt{(x+y)(y-x^2+1)}}$

Soln:
$(x+y)(y-x^2+1) > 0$

There are 2 cases for this:
1. $(x+y) > 0$ and $(y-x^2+1) > 0$
   $y > -x$ $\quad$ $y > x^2-1$

2. $(x+y) < 0$ and $(y-x^2+1) < 0$
   $y < -x$ $\quad$ $y < x^2-1$